Question 1
Part a
There are 4 loops in the function with indexes i, j, k, and l, let’s call these loops I, J, K and L respectively. Clearly loop I executes $n$ times. Let’s make a table for the number of executions of the loops for all values of index $i$, (ignoring the conditional statement):

| $i$ | $|J|$ | $|K|$ | $|L|$ | $|J||K|$ |
|-----|------|------|------|--------|
| 0   | $n$  | $\lfloor \log(n) \rfloor + 1$ | $n$  | $n(\lfloor \log(n) \rfloor + 1)$ |
| 1   | $n-1$| $\lfloor \log(n) \rfloor + 1$ | $n-1$| $(n-1)(\lfloor \log(n) \rfloor + 1)$ |
| 2   | $n-2$| $\lfloor \log(n) \rfloor + 1$ | $n-2$| $(n-2)(\lfloor \log(n) \rfloor + 1)$ |
| ... | ...  | ...  | ...  | ...    |
| n-1 | 1    | $\lfloor \log(n) \rfloor + 1$ | 1    | $\lfloor \log(n) \rfloor + 1$ |

Whenever the $x_i = 1$, loops J and K are executed, otherwise loop L is executed. It can be seen that whenever the $x_i = 1$, the algorithm executes more operations. Therefore for the best case analysis the input $X$ is a stream of all zeros, whereas for the worst case, $X$ is a stream of all ones.

Best Case:
Input $X$ is:

$$X = [1 \ 1 \ \ldots \ 1]$$

The number of executions:

$$B(n) = \sum_{i=0}^{n-1} (n-i) = n^2 - \sum_{i=0}^{n-1} i = n^2 - \frac{n(n-1)}{2}$$

$$= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\in \Theta(n^2)$$
Worst Case:

Input \( X \) is:

\[
X = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}
\]

The number of executions:

\[
W(n) = \sum_{i=0}^{n-1} (n-i)(\lfloor \log(n) \rfloor + 1) = (\log(n) + 1) \sum_{i=0}^{n-1} (n-i) = \frac{n(n+1)(\lfloor \log(n) \rfloor + 1)}{2}
\]

\[
= \frac{1}{2}n^2(\lfloor \log(n) \rfloor + 1) + \frac{1}{2}n(\lfloor \log(n) \rfloor + 1)
\]

\[\in \Theta(n^2 \log(n))\]

Average Case:

We assume that the input is uniform. Which means \( p(x_i = 0) = \frac{1}{2} \) and \( p(x_i = 1) = \frac{1}{2} \). Then, the average number of executions per step becomes:

\[
\frac{1}{2}(n-i) + \frac{1}{2}(n-i)(\lfloor \log(n) \rfloor + 1)
\]

Therefore,

\[
A(n) = \sum_{i=0}^{n-1} \frac{1}{2}((n-i) + (n-i)(\lfloor \log(n) \rfloor + 1))
\]

\[
= \frac{1}{2}\left(\frac{n(n+1)}{2} + \frac{n(n+1)(\lfloor \log(n) \rfloor + 1)}{2}\right)
\]

\[= \frac{1}{4}(n^2 + n)(\lfloor \log(n) \rfloor + 2)
\]

\[\in \Theta(n^2 \log(n))\]

Question 2

We use Stirling’s approximation to simplify the term \( \log(2n!) \).

\[n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n\]

The function becomes:

\[f(n) \approx n^2 \log \left(2\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right) + 3n^3 + \sqrt{n}\]

\[= n^2 \left(\log 2 + \frac{1}{2} \log(2\pi n) + n \log(n) - n \log(e)\right) + 3n^3 + \sqrt{n}\]

\[= n^3 \log(n) + 3n^3 + n^3 \log(e) + \frac{1}{2}n^2 \log(2\pi n) + n^2 + \sqrt{n}\]

The function has 6 terms, all positive and monotonically increasing.
a) \( f(n) \in O(n^3) \): False

We can directly show that there is no \( c \) and \( n_0 \) such that \( f(n) \leq cn^3 \). Let’s look at the term \( n^3 \log(n) \).

\[
n^3 \log(n) \leq cn^3, \forall n \geq n_0
\]

\( \log(n) \) is a monotonically increasing function and \( c \) is constant. For all \( \{c, n_0\} \) pairs, there is an \( n \) value, which is greater than \( n_0 \), which makes \( \log(n) > c \). We don’t need to look at other terms.

b) \( f(n) \in o(n^3 \log(n)) \): False

We use the definition of little-oh:

\[
f(n) \in o(g(n)) \rightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0
\]

\[
\lim_{n \to \infty} \frac{f(n)}{n^3 \log(n)} = \lim_{n \to \infty} \frac{n^3 \log(n)}{n^3 \log(n)} + \lim_{n \to \infty} \frac{3n^3}{n^3 \log(n)} + \lim_{n \to \infty} \frac{n^3 \log(e)}{n^3 \log(n)}
\]

\[
+ \lim_{n \to \infty} \frac{\frac{1}{2}n^2 \log(2\pi n)}{n^3 \log(n)} + \lim_{n \to \infty} \frac{n^2}{n^3 \log(n)} + \lim_{n \to \infty} \frac{\sqrt{n}}{n^3 \log(n)}
\]

\[
= \lim_{n \to \infty} \frac{n^3 \log(n)}{n^3 \log(n)} = 1
\]

therefore, \( f(n) \notin o(n^3 \log(n)) \).

c) \( f(n) \in \Theta(n^3 \log(n)) \): True

We can find \( c_1, c_2 \) and \( n_0 \) such that

\[
c_1(n^3 \log(n)) \leq f(n) \leq c_2(n^3 \log(n)) \quad \forall n \geq n_0
\]

The function \( f(n) \) has 6 terms and with \( n^3 \log(n) \) as the greatest order. Let \( n_0 = 8 \). Then,

\[
3n^3 \leq n^3 \log(n)
\]

\[
n^3 \log(e) \leq n^3 \log(n)
\]

\[
\frac{1}{2}n^2 \log(2\pi n) \leq n^3 \log(n)
\]

\[
n^2 \leq n^3 \log(n)
\]

\[
\sqrt{n} \leq n^3 \log(n)
\]

for all \( n \geq n_0 \). Therefore, we can let \( c_2 = 6 \). An obvious choice for \( c_1 \) is 1. It is clear that for all \( n \geq n_0 = 8 \) the condition holds. Therefore \( f(n) \in \Theta(n^3 \log(n)) \).
d) \( f(n) \in \Omega(n^3) \): True

It is very easy to show directly. If we can find \( c \) and \( n_0 \) such that \( cg(n) \leq f(n) \) for all \( n \geq n_0 \), then \( f(n) \in \Omega(n^3) \). Consider the term \( n^3 \log(n) \):

\[
\begin{align*}
  cn^3 & \leq n^3 \log(n) \\
  c & \leq n^3 \log(n)
\end{align*}
\]

Let \( c = 1 \). \( \log(n) \geq 1 \) for all values of \( n \geq 2 \) (\( n_0 = 2 \)). This also implies that \( cg(n) \leq f(n) \) since the remaining terms of \( f(n) \) are positive. The proof is completed.

**Remark 1:** You cannot just give numbers to \( c_1, c_2 \) and \( n_0 \). You have to justify why you choose them.

**Remark 2:** You can not prove that the necessary condition holds for a single assignment of constants. You have show that for all \( n > n_0 \) the condition holds.